Knowledge Organiser: Mathematics
Year 12 – Spring Term 2

Suggested websites: TL Maths and Maths Watch

Key point

If $f(x) = ax^n$ then $f'(x) = nax^{n-1}$

Key point

If $y = f(x)$ then $\frac{dy}{dx} = f'(x)$

Unit 6: Differentiation (PURE)

- 6a. Definition, differentiating polynomials, second derivatives
- 6b. Gradients, tangents, normals, maxima and minima

Key Vocabulary

Differentiation, derivative, first principles, rate of change, rational, constant, tangent, normal, increasing, decreasing, stationary point, maximum, minimum, integer, calculus, function, parallel, perpendicular.

Worked example

Given that $y = 3x^6 - 8 + \frac{1}{x^3}, x \neq 0$, find $\frac{dy}{dx}$ (3 marks)

$y = 3x^6 - 8 + x^{-3}$

$\frac{dy}{dx} = 18x^5 - 3x^{-4}$

Differentiation 1

You can **differentiate** a function to find its **derivative** or **gradient function**.

The derivative is written as $f'(x)$ or $\frac{dy}{dx}$

Differentiating x^n

$y = f(x)$ $\quad$ $y = x^n$ — Multiply by the power ...

Differentiation

$\frac{dy}{dx} = f'(x)$ $\quad$ $\frac{dy}{dx} = nx^{n-1}$ ... then reduce the power by 1

This rule works for **any** value of n, including fractions and negative numbers.

Golden rules

1. Write every term in a polynomial in the form ax^n **before** differentiating.
   $\sqrt{x} \to x^{\frac{1}{2}}$ $\quad$ $\frac{6}{x^2} \to 6x^{-2}$
2. Constant terms differentiate to **zero**, and x terms differentiate to **a constant**.
   $f(x) = 7 \to f'(x) = 0$ $\quad$ $f(x) = 3x + 1 \to f'(x) = 3$

Evaluating $f'(x)$

$f'(x)$ tells you the **rate of change** of the function for a given value of x.
You can calculate $f'(x)$ for a given value of x by substituting that value of x into the derivative.

Second-order derivatives

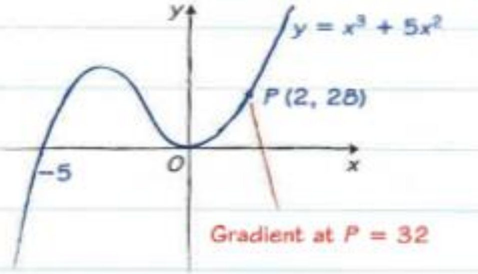
You can differentiate twice to find the **second-order** derivative.
You write the second-order derivative as $\frac{d^2y}{dx^2}$ or $f''(x)$.

$y = 5x^3$ → Differentiate → $\frac{dy}{dx} = 15x^2$ → Differentiate → $\frac{d^2y}{dx^2} = 30x$

Differentiation 2

You can use the **derivative** or **gradient function** to find the **rate of change** of a function, or the gradient of a curve.

This curve has equation $y = x^3 + 5x^2$. Its gradient function has equation $\frac{dy}{dx} = 3x^2 + 10x$. You can find the **gradient** at any point on the graph by substituting the x-coordinate at that point into the gradient function.



At the point P:
$x = 2$
$\frac{dy}{dx} = 3(2)^2 + 10(2) = 12 + 20 = 32$

Worked example

Given that $y = 8\sqrt{x} - 3x^2 + 5x, x > 0$, find $\frac{d^2y}{dx^2}$ (4 marks)

$y = 8x^{\frac{1}{2}} - 3x^2 + 5x$

$\frac{dy}{dx} = 4x^{-\frac{1}{2}} - 6x + 5$

$\frac{d^2y}{dx^2} = -2x^{-\frac{3}{2}} - 6$

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Key point
Integrating x^n with respect to x is written as $\int x^n dx = \frac{x^{n+1}}{n+1} + c, n \neq -1$

Unit 7: Integration (PURE)

- 7a. Definition as opposite of differentiation, indefinite integrals of x^n
- 7b. Definite integrals and areas under curves

Key Vocabulary

Calculus, differentiate, integrate, reverse, indefinite, definite, constant, evaluate, intersection.

Integration

Integration is the **opposite** of differentiation. You can use this rule to integrate terms which are written in the form ax^n.

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c, n \neq -1$$

- This is the symbol for integration.
- You **increase** the power by 1 …
- … then **divide** by the **new** power.
- This rule **doesn't** work if the original power is −1.
- You are **integrating with respect to** x.
- You have to add **the constant of integration.**

To **integrate** a function, write each term in the form ax^n, then integrate one term at a time.

The constant of integration

When you **differentiate**, any **constant terms** disappear. So lots of functions have the same derivative.

$y = x^2 + 5$, $y = x^2$, $y = x^2 - 19$ → Differentiate → $\frac{dy}{dx} = 2x$ → Integrate → $y = x^2 + c$

When you integrate you don't know the constant. You write '+ c' at the end to show this. This is called **indefinite integration.**

Golden rules

1. Write every term in a polynomial in the form ax^n **before** integrating.
2. Remember to include the **constant of integration.**
3. Simplify any **coefficients** if possible.

Worked example

Given that $y = \frac{1}{x^3} - 3x^5, x \neq 0$, find $\int y\,dx$ **(3 marks)**

$\int (x^{-3} - 3x^5)\,dx = \frac{x^{-2}}{-2} - \frac{3x^6}{6} + c$

$= -\frac{1}{2}x^{-2} - \frac{1}{2}x^6 + c$

Be careful with negative powers. For the first term, you have to **increase** the power of −3 by 1 to get −2, then divide by the **new** power, −2.

Worked example

Find $\int(12x^3 + 6x - 15x^{\frac{2}{3}})\,dx$, giving each term in its simplest form. **(5 marks)**

$\int(12x^3 + 6x - 15x^{\frac{2}{3}})\,dx$

$= \frac{12x^4}{4} + \frac{6x^2}{2} - \frac{15x^{\frac{5}{3}}}{\left(\frac{5}{3}\right)} + c$

$= 3x^4 + 3x^2 - 9x^{\frac{5}{3}} + c$

Integrate term-by-term and don't forget to add the constant of integration.
For each term:
- increase the power by 1
- divide by the **new** power.

$\frac{2}{3} + 1 = \frac{5}{3}$. Dividing by $\frac{5}{3}$ is the same as dividing by 5 then multiplying by 3.

When you perform an integration you can check your result by differentiating it – you should get back to what you started with.

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Key point: The **null hypothesis**, H_0, is a statistical statement representing your basic assumption.

Key point: The **alternative hypothesis**, H_1, is a statement that contradicts the null hypothesis.

Unit 5a: Statistical hypothesis testing (Stats)

5a. Language of hypothesis testing; Significance levels

Key Vocabulary

. Hypotheses, significance level, one-tailed test, two-tailed test, test statistic, null hypothesis, alternative hypothesis, critical value, critical region, acceptance region, p-value, binomial model, accept, reject, sample, inference.

Hypothesis testing

You need to be able to carry out a hypothesis test for the probability, p, in a binomial distribution. Follow these steps to carry out a hypothesis test.

Model the test statistic and define null (H_0) and alternative (H_1) hypotheses. → Assume H_0 is true and calculate the probability of the observed value (or a greater / lesser value) occurring → Compare this probability with a given significance level and write a conclusion stating whether H_0 is accepted or rejected.

Worked example

A microchip manufacturer knows that 9% of the microchips produced using a certain process contain defects. The manufacturer trials a new manufacturing process. A sample of 50 chips from the new process are selected and 2 of them are observed to be faulty.

Test, at the 10% significance level, whether there is evidence that the proportion of faulty chips has reduced under the new process. State your hypotheses clearly. **(6 marks)**

Let X = the number of faulty chips in a sample of 50. Then $X \sim B(50, p)$.

$H_0: p = 0.09$, $H_1: p < 0.09$

Assume H_0 is true, so $X \sim B(50, 0.09)$.

$P(X \leqslant 2) = 0.1605...$

16% > 10% so there is not enough evidence to reject H_0.

The proportion of faulty chips has not significantly reduced under the new process.

How many tails?

✓ If you want to test whether p is likely to be **greater than** or **less than** a particular value you need to use a one-tailed test. For example:

$H_0: p = 0.4$, $H_1: p > 0.4$

✓ If you want to test whether p is likely to be **different** from a particular value, you need to use a two-tailed test. For example:

$H_0: p = 0.75$, $H_1: p \neq 0.75$

Problem solved!

You want to test whether the proportion has **reduced** so this is a **one-tailed test**.

You can use your calculator to find $P(X \leqslant 2)$ directly. Since the probability of this observation (or worse) is **greater than** 10% you **do not** reject H_0.

You will need to use problem-solving skills throughout your exam – **be prepared!**

Actual significance level

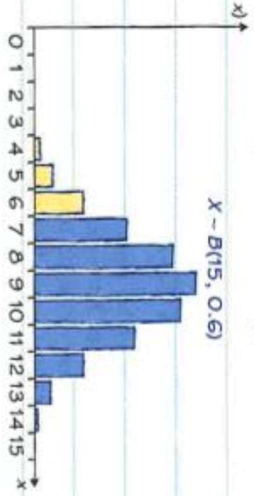
The actual probability that the observed value will fall within the critical region is sometimes called the actual significance level. This is also the probability that the null hypothesis is **rejected incorrectly.**

Critical regions

In a hypothesis test, you can find the observed values of a random variable which would cause you to reject the null hypothesis. This set of values is called the **critical region.**

1 One-tailed tests

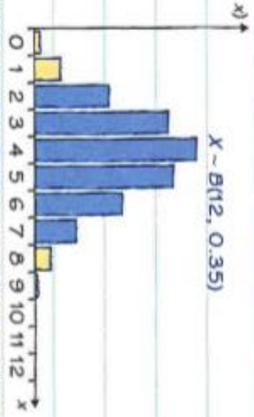
For $X \sim B(15, p)$, if you are testing $H_0: p = 0.6$ against $H_1: p < 0.6$ at the 10% level, the critical region is $X \leqslant 6$.



$P(X \leqslant 6) = 0.09505$, which is the first value of x with $P(X \leqslant x) < 0.1$. The value 6 is called the critical value.

2 Two-tailed tests

For $X \sim B(12, p)$, if you are testing $H_0: p = 0.35$ against $H_1: p \neq 0.35$ at the 10% level, the critical region is $X \leqslant 1$ and $X \geqslant 8$.



Divide the significance level by 2. The probability in **each tail** should be $\leqslant 0.05$. $P(X \leqslant 1) = 0.04244$ and $P(X \geqslant 8) = 0.02551$. There are two **critical values**: 1 and 8.

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Key point If a resultant force **F** N acts on an object of mass m kg giving it an acceleration **a** m s^{-2} then **F** = m**a**

Key point If forces $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$ act on an object then the resultant force is $\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \ldots \mathbf{F}_n$

Unit 8b: Forces & Newton's laws (Mechanics)

8b. Newton's second law, (no resolving forces or use of F = μR); Newton's third law: equilibrium, smooth pulley problems

Key Vocabulary

Force, newtons, mass, weight, gravity, tension, thrust, compression, air resistance, reaction, driving force, braking force, resultant, force diagram, equilibrium, inextensible, light, negligible, particle, smooth, uniform, pulley, string, retardation, free particle.

Forces

A force acting on an object has **direction** and **magnitude**. The units of force are **newtons** (N). 1 newton is the force needed to accelerate a 1 kg object at a rate of 1 m s^{-2}. Because of this, the units of force can be written as kg m s^{-2}.

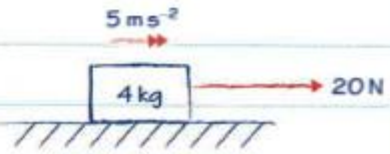
$F = ma$

$F = ma$ is sometimes called the **equation of motion**. In words it is:

force (N) = mass (kg) × acceleration (m s^{-2})

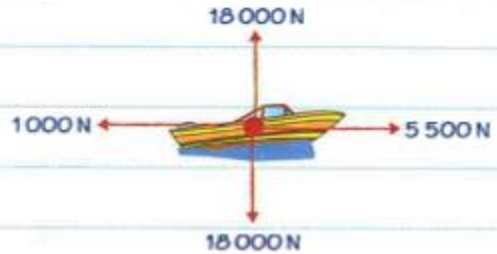
You need to remember $F = ma$. It is not in the formulae booklet.

This 4 kg block is resting on a smooth surface. If it is acted on by a force of 20 N it will accelerate at a rate of 5 m s^{-2}.



Resultant force

If there is more than one force acting on a particle you can find the **resultant** in any given direction.



This boat is accelerating. The vertical forces have the same magnitude so their resultant is **zero**. The resultant force in the horizontal direction is 5500 − 1000 = 4500 N.

Key point Newton's third law states that when an object A exerts a force on an object B, object B exerts an equal and opposite force on object A.

Examples of forces include:

Kicking a ball with a force of magnitude 200 N in the easterly direction.

A force $\mathbf{F} = (3\mathbf{i} + 4\mathbf{j})$N acting on a particle.

Strategy

To solve questions involving acceleration

1. Draw a clear diagram, marking on all the forces which act on the object and the acceleration.
2. Use $F = ma$ to write an equation of motion where F is the sum of the components of all the forces in the direction of a.
3. Solve the equation to calculate the unknown force.

Example 1

Calculate the acceleration if the forces acting on an object of mass 25 kg are $(40\mathbf{i} + 15\mathbf{j})$ N, $(20\mathbf{i} - 7\mathbf{j})$ N and $(31\mathbf{i} + 23\mathbf{j})$ N.

The resultant force $\mathbf{F} = (40\mathbf{i}+15\mathbf{j}) + (20\mathbf{i}-7\mathbf{j}) + (31\mathbf{i}+23\mathbf{j}) = 91\mathbf{i} + 31\mathbf{j}$ N

$91\mathbf{i} + 31\mathbf{j} = 25\mathbf{a}$

$\therefore \mathbf{a} = (3.64\mathbf{i} + 1.24\mathbf{j})$ m s^{-2}

Use $\mathbf{F} = m\mathbf{a}$

Remember that acceleration is a vector not a scalar.

Key point If a resultant force **F** N acts on an object of mass m kg giving it an acceleration **a** m s^{-2} then **F** = m**a**